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RATIONAL SURFACES DEFINED BY LINEAR SYSTEMS OF PLANE CURVES $C_{3n}: 8A^nB^{n-1}$

A Thesis

PRESENTED TO THE GRADUATE SCHOOL OF CORNELL UNIVERSITY IN CANDIDACY FOR THE DEGREE OF DOCTOR OF PHILOSOPHY

1929

BY

JOSEPH CRAWFORD POLLEY

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Rational Surfaces Defined by Linear Systems of Plane Curves $C_{3n}: 8A^nB^{n-1}$.

By Joseph Crawford Polley.

1. *Introduction.* The rational surfaces of order four and five having no multiple curves, and those of order five having multiple curves insufficient for rationality, have been determined. There are three types of rational quartic surfaces with no double curve. One was discovered by L. Cremona * by applying a cremona transformation to a known quartic surface. The two remaining types were determined by M. Noether.† His method of investigation was that of considering quartic surfaces with a double point and applying to their equations the conditions for a one to one correspondence with the points of a plane.

Of particular interest also is the work of D. Montesano on rational quintic surfaces.‡ He obtained all the possible types by applying special cremona transformations to known rational surfaces.

In this paper various rational surfaces are discussed by considering certain linear systems of plane curves. The surfaces of Cremona, Noether and some of those of Montesano are re-determined by this method and a general type of surface is discussed by means of a linear system of plane curves of the form $C_{3n}: 8A^nB^{n-1}$.

2. *The system* $C_6: 7A^2$. In a plane (x) the system of curves $C_6: 7A^2$, with double points at 7 points A_i $(i = 1, 2, \cdots, 7)$, is of dimension 6. Let $C_3: 7A$, $C_3': 7A$ and $C_3'': 7A$ be linearly independent members of the net of cubics determined by the points A_i; and $C_6: 7A^2$ a non-composite sextic of the system. Then we can take as the equation of the system

$$(1)\quad a_1C_3^2 + a_2C_3C_3' + a_3C_3C_3'' + a_4C_3'C_3'' + a_5C_3'^2 + a_6C_3''^2 + a_7C_6 = 0.$$

Let

$$(2)\quad \begin{aligned} &y_1 = C_3^2 : 7A, \quad y_2 = C_3 : 7A \cdot C_3' : 7A, \quad y_3 = C_3 : 7A \cdot C_3'' : 7A, \\ &y_4 = C_3' : 7A \cdot C_3'' : 7A, \quad y_5 = C_3'^2 : 7A, \quad y_6 = C_3''^2 : 7A, \quad y_7 = C_6 : 7A^2. \end{aligned}$$

* L. Cremona, *Coll. Math. Chelini* 413-424 (1881).

† M. Noether, *Mathematische Annalen*, Vol. 33 (1889), pp. 546-571.

‡ D. Montesano, *Rendiconti della Reale Accademia di Napoli*, Ser. 3, Vol. 13 (1907), pp. 66-68.

These are the parametric equations of a rational surface F_a in S_6, in $(1,1)$ correspondence with the plane (x), the image of a point (x) being a point (y) on the surface. Since any two members of the set have eight residual intersections, a general S_4 in S_6 meets F_a in eight points. Hence F_a is of order eight.

For a general point on C_3

$$(3) \quad y_1 = y_2 = y_3 = 0, \quad y_4 = C_3'C_3'', \quad y_5 = C_3'^2, \quad y_6 = C_3''^2, \quad y_7 = C_6.$$

Hence the image of C_3 is a curve L in a sub-space S_3 of S_6.

Since a $C_6 : 7A^2$ has, with $C_3 : 7A$, four residual intersections, a plane

$$y_1 = y_2 = y_3 = \beta_4 y_4 + \beta_5 y_5 + \beta_6 y_6 + \beta_7 y_7 = 0$$

meets L in four points. Hence L is a quartic curve of genus 1.

By projection from the plane of $y_1 = y_2 = y_3 = y_7 = 0$ the surface F_8 goes into a surface F_4, in an S_3, whose parametric equations are

$$(4) \quad \begin{aligned} &y_1 = C_3^2 : 7A, && y_2 = C_3 : 7A \cdot C_3' : 7A, \\ &y_3 = C_3 : 7A \cdot C_3'' : 7A, && y_7 = C_6 : 7A^2. \end{aligned}$$

The image of L is the point $(0,0,0,1)$. To the plane sections of F_4 corresponds the system

$$a_1 C_3^2 + a_2 C_3 C_3' + a_3 C_3 C_3'' + a_4 C_6 = 0$$

all members of which pass through the four simple points in which C_3 meets C_6. The system is therefore of grade 4.

For a point near P_i $(i = 1, 2, 3, 4)$, the residual intersections of C_3 and C_6, the corresponding y_2, y_3 and y_7 are infinitesimals of the first order and the corresponding y_1 is an infinitesimal of the second order. Hence the images of P_i are straight lines in the plane $y_1 = 0$ through the point $(0,0,0,1)$.

The surface (4) is the well known rational quartic surface of Cremona.

3. *The system* $C_7 : A^3 8B^2$. In a plane (x) the system of curves $C_7 : A^3 8B^2$, with a triple point at A and double points at B_i $(i = 1, 2, \cdots, 8)$, is of dimension 5. The basis points determine a cubic $C_3 : A8B$ and a web of quartics $C_4 : A^2 8B$. Hence, if $C_7 : A^3 8B^2$ and $C_7' : A^3 8B^2$ are two non-composite curves of the system, and $C_1 : A$, $C_1' : A$ are members of the pencil of lines on A, the following can be taken as the linearly independent members of the system:

$$(5) \quad \begin{aligned} &C_7 : A^3 8B^2, && C_3 : A8B \cdot C_4 : A^2 8B, && C_3^2 : A8B \cdot C_1 : A, \\ &C_3^2 : A8B \cdot C_1' : A, && C_3 : A8B \cdot C_4' : A^2 8B, && C_7' : A^3 8B^2. \end{aligned}$$

A member of the pencil $C_7 + \gamma C_7' = 0$ has two residual intersections with C_3 and for a particular choice of γ there is a member tangent to C_3 at some point C. Let C_7' be that member and consider the pencil $C_7' + \delta C_3 C_4' = 0$. For a particular choice of δ we obtain a C_7 with a double point at C; call it $\bar{C}_7$. Furthermore, among the quartics there is a net through the point C, $C_4 : A^2 8BC$, $C_3 : A8BC \cdot C_1 : A$ and $C_3 : A8BC \cdot C_1' : A$. Let D be the residual intersection of C_4 and C_3. Then we have as linearly independent members of the system

$$(6)\quad \begin{array}{lll} \bar{C}_7 : A^3 8B^2C^2, & C_3 : A8BC \cdot C_4 : A^2 8BCD, & C_3^2 : A8BCD \cdot C_1 : AD, \\ C_3^2 : A8BCD \cdot C_1' : A, & C_3 : A8BCD \cdot C_4' : A^2 8B, & C_7' : A^3 8B^2. \end{array}$$

Let

$$(7)\quad y_1 = \bar{C}_7,\ y_2 = C_3 \cdot C_4,\ y_3 = C_3^2 \cdot C_1,\ y_4 = C_3^2 \cdot C_1',\ y_5 = C_3 \cdot C_4',\ y_6 = C_7',$$

and project from the line $y_1 = y_2 = y_3 = y_4 = 0$ into the opposite S_3 of the S_5, thus obtaining a surface whose parametric equations are

$$(8)\quad \begin{array}{ll} y_1 = C_7 : A^3 8B^2C^2, & y_2 = C_3 : A8BCD \cdot C_4 : A^2 8BCD, \\ y_3 = C_3^2 : A8BCD \cdot C_1 : 8AD, & y_4 = C_3^2 : A8BCD \cdot C_1 : 8A. \end{array}$$

The surface defined by equations (8) is a rational quartic surface F_4 since the system (8) is of grade 4.

For a general point on $C_3 : A8BCD$, $y_1 \neq 0$ and $y_2 = y_3 = y_4 = 0$, hence the image of $C_3 : A8BCD$ is the point $(1, 0, 0, 0)$ on F_4. The section of F_4 made by a plane $ky_2 + ly_3 = my_4 = 0$ through the point $(1, 0, 0, 0)$ determines in the plane (x) a composite curve

$$C_3 : A8BCD(kC_4 : A^2 8BCD + lC_3 : A8BCD \cdot C_1 : AD + mC_3 : A8BCD \cdot C_1' : A) = 0$$

that is

$$(9)\qquad C_3 : A8BCD \cdot C_4 : A^2 8BCD.$$

Any two curves of form (9) have two residual intersections. Therefore $(1, 0, 0, 0)$ is a double point. Since to each plane section through $(1, 0, 0, 0)$ corresponds a single C_4, hence but one direction through the point D in (x), any plane section through $(1, 0, 0, 0)$ has a cusp at that point.

Let a point P in (x) approach a point Q, other than D, on C_3; then y_3 and y_4 vanish to the second order, y_2 to the first order, and y_1 is finite; hence the image of P approaches $(1, 0, 0, 0)$ along $y_3 = y_4 = 0$, and $y_3 = y_4 = 0$ is the cuspidal tangent. As a point P approaches D on C_3, y_3 vanishes to the

third order, hence $(1, 0, 0, 0)$ is a uniplanar singular point on F_4, the plane of the point being the plane $y_3 = 0$.

The surface (8) is one of the rational quartic surfaces with no double curve, determined by Noether.

4. *The system* $C_9 : 8A^3B^2$. In a plane (x) the system of curves $C_9 : 8A^3B^2$ with triple points at A_i $(i = 1, 2, \cdots, 8)$ and a double point at B is of dimension 4. Taking $C_3 : 8AB$ and $C_3 : 8A$ as members of the pencil of cubics on A_i; $C_6 : 8A^2B$, a non-composite sextic; and $C_9 : 8A^3B^2$ a non-composite curve of order 9, the equation of the system is

$$(10) \qquad a_1C_9 : 8A^3B^2 + a_2C_3 : 8AB \cdot C_6 : 8A^2B + a_3C_3^2 : 8AB \cdot C_3 : 8A + a_4C_3^3 : 8AB = 0.$$

The cubic C_3 has with C_9 one residual intersection. Call this point C. Let

$$(11) \qquad \begin{aligned} y_1 &= C_9 : 8A^3B^2C, & y_2 &= C_3 : 8ABC \cdot C_6 : 8A^2B, \\ y_3 &= C_3^2 : 8ABC \cdot C_3 : 8A, & y_4 &= C_3^3 : 8ABC. \end{aligned}$$

These are the parametric equations of a rational surface of order 4 in ordinary space. The image of C_3 is the point $(1, 0, 0, 0)$.

For a point near C in (x) the corresponding values of y_1 and y_2 are infinitesimals of the first order and those of y_3 and y_4 are infinitesimals of the second order and the third order respectively. Hence the image of C is the line $y_3 = y_4 = 0$.

For a point near B in (x) the corresponding values of y_1, y_2 and y_3 are infinitesimals of the second order and that of y_4 is an infinitesimal of the third order. Hence the image of B is a conic in the plane $y_4 = 0$.

There is a pencil of curves $C_9 : 8A^3B^2C^2$ given by

$$C_3^2 : 8ABC \cdot C_3 : 8A + \gamma C_3^3 : 8ABC = 0.$$

These go into the sections of F_4 cut by the pencil of planes $y_4 = \gamma y_3$, these sections being composed of rational cubics and the line $y_3 = y_4 = 0$. To the section made by the plane $y_4 = 0$ corresponds $C_3^3 : 8ABC$ which is of the form $C_9 : 8A^3B^3C^3$, hence the section made by this plane is composed of a conic, image of B, and a line image of C, taken twice.

This surface is the second rational quartic surface with no double point determined by Noether.

5. *The system* $C_{12} : 8A^4B^3$. In a plane (x) the system of curves $C_{12} : 8A^4B^3$ is of dimension 4 and the equation of the system may be written

(12) $a_1C_{12}: 8A^4B^3D + a_2C_3: 8ABD \cdot C_9: 8A^3B^2 + a_3C_3^2: 8ABD \cdot C_6: 8A^2B + a_4C_3^3: 8ABD \cdot C_3: 8A + a_5C_3^4: 8ABD = 0$

where D is the residual intersection of $C_3: 8AB$ and $C_{12}: 8A^4B^3$.

Let

(13) $y_1 = C_{12}: 8A^4B^3D$, $y_2 = C_3: 8ABD \cdot C_9: 8A^3B^2$,
$y_3 = C_3^2: 8ABD \cdot C_6: 8A^2B$, $y_4 = C_3^3: 8ABD \cdot C_3: 8A$, $y_5 = C_3^4: 8ABD$.

These are the parametric equations of a rational F_6 in S_4.

Let E be the residual intersection of $C_{12}: 8A^4B^3D = 0$ and $C_9: 8A^3B^2 = 0$. Then through E passes a curve of the pencil $C_3: 8A$ and a non-composite curve of the net $C_6: 8A^2B$. The image of E is a point P on F_6. Project the F_6 from $(0, 0, 0, 0, 1)$ as a center into the opposite S_3 obtaining an F_5 whose parametric equations are

(14) $y_1 = C_{12}: 8A^4B^3DE$, $y_2 = C_3: 8ABD \cdot C_9: 8A^3B^2E$,
$y_3 = C_3^2: 8ABD \cdot C_6: 8A^2BE$, $y_4 = C_3^3: 8ABD \cdot C_3: 8AE$.

Each C_{12} of the system goes into a plane section of F_5 which is a C_5 of genus 4, hence a C_5 with two double points. The locus of these double points must be a double conic K on F_5.

The image of point D is the line $y_3 = y_4 = 0$; the image of point E is a line on the surface.

(15) The first polar of a rational surface F_N with respect to a point P not on the surface is a surface of order $N - 1$, containing the double curve on F_N, the curve of contact of the tangent cone to F_N with P as a vertex, and the singular points on F_N. If F_N is a surface whose plane sections are mapped on a plane (x) by a web of curves of order m, then corresponding to the plane sections of F_N through P is a net of curves belonging to the web, whose Jacobian is a curve of order $3(m-1)$ having a $(3r-1)$-fold point at an r-fold point of the web. The Jacobian is the image of the curve of contact of the tangent cone with F_N.

For the case at hand the polar surface is an F_4 whose intersection with F_5 goes into a composite $C_{48}: 8A^{16}B^{12}D^4E^4$ consisting of the Jacobian $C_{33}: 8A^{11}B^8D^2E^2$; the curve $C_3: 8ABD$, image of $(0, 0, 0, 1)$; and the image of the double conic K which is a $C_{12}: 8A^4B^3DE^2$. A general $C_{12}: 8A^4B^3DE$ goes into a general plane section of F_5 but the $C_{12}: 8A^4B^3DE^2$ goes into the double conic counted twice and a residual line of F_5 in the plane of the conic. This line is the image of point E.

5(a). If point B is chosen on a certain locus there is a $C_9: 8A^3B^3$ other than

$C_3^3 : 8AB$.* Hence we can choose as linearly independent members of the system

$$(16)\quad \begin{array}{lll} C_9 : 8A^3B^3 \cdot C_3 : 8ABC, & C_9 : 8A^3B^3 \cdot C_3 : 8AC, & \\ C_3^3 : 8ABC \cdot C_3 : 8AC, & C_3^2 : 8ABC \cdot C_6 : 8A^2B, & C_3^4 : 8ABC, \end{array}$$

where C is the ninth point common to $C_3 : 8ABC$ and $C_3 : 8A$.

Take any point D in the plane (x). There is a member of the pencil $C_9 : 8A^3B^3 + \gamma C_3^3 : 8ABC = 0$ through D. Call this a new $C_9 : 8A^3B^3$. There is also a $C_3 : 8A$ and a $C_6 : 8A^2B$ through D.

Let

$$(17)\quad \begin{array}{ll} y_1 = C_9 : 8A^3B^3D \cdot C_3 : 8ABC, & y_2 = C_9 : 8A^3B^3D \cdot C_3 : 8ACD, \\ y_3 = C_3^2 : 8ABC \cdot C_6 : 8A^2BD, & y_4 = C_3^3 : 8ABC \cdot C_3 : 8ACD. \end{array}$$

These are the parametric equations of a rational surface F_5 in S_3.†

The image of $C_9 : 8A^3B^3D$ is the line $y_1 = y_2 = 0$. The image of $C_3 : 8ACD$ is the line $y_2 = y_4 = 0$. The image of the point C is the line $y_3 = y_4 = 0$, the image of point D a line in the plane $y_2 = 0$, and the image of point B a rational cubic in the plane $y_1 = 0$, with a double point at $(0, 1, 0, 0)$.

Since C_9 has two residual intersections with a general C_{12}, each plane section of F_5 has a double point on the line $y_1 = y_2 = 0$, the image of C_9. Hence $y_1 = y_2 = 0$ is a double line.

A section of F_5 made by a plane of the pencil $y_2 = \gamma y_4$ goes into a composite C_{12} of the form $C_3 : 8ACD \cdot C_9 : 8A^3B^3$, which meets a general C_{12} in 3 residual points not on $C_3 : 8ACD$. Hence the line of intersection of any plane of the pencil and a plane not of the pencil meets F_5 in 3 points other than on the line $y_2 = y_4 = 0$, which means that $y_2 = y_4 = 0$ is a double line.

We observe that the surface F_5 as defined by (17) has a composite double conic consisting of the double lines $y_1 = y_2 = 0$ and $y_2 = y_4 = 0$. As in the general case (14), the image line of the point D is the residual intersection of the plane of the double conic with F_5.

6. *The system* $C_{15} : 8A^5B^4$. In a plane (x) the system of curves $C_{15} : 8A^5B^4$ is of dimension 5 and by the method employed in the previous cases we can choose as linearly independent members of the system

$$(18)\quad \begin{array}{lll} C_{15} : 8A^5B^4, & C_{12} : 8A^4B^3 \cdot C_3 : 8AB, & C_9 : 8A^3B^2 \cdot C_3^2 : 8AB, \\ C_6 : 8A^2B \cdot C_3^3 : 8AB, & C_3^4 : 8AB \cdot C_3 : 8A, & C_3^5 : 8AB. \end{array}$$

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* Halphen, *Bulletin de la So. Math.*, 162 (1882).

† D. Montesano, *Rendiconti della Reale Accodemia di Napoli*, Ser. 3, Vol. 13 (1907), pp. 66-68.

Let C be the residual intersection of $C_3: 8AB$ and $C_{15}: 8A^5B^4$. The curves $C_{15}: 8A^5B^4$ and $C_{12}: 8A^4B^3$ have 8 residual points of intersection. Call two of these points D and E. Choose a new C_9 and a new C_6 such that they will pass through D and E.

Let

$$(19)\quad \begin{aligned} y_1 &= C_{15}: 8A^5B^4CDE, & y_2 &= C_{12}: 8A^4B^3DE \cdot C_3: 8ABC,\\ y_3 &= C_9: 8A^3B^2DE \cdot C_3{}^2: 8ABC, & y_4 &= C_6: 8A^2BDE \cdot C_3{}^3: 8ABC,\\ y_5 &= C_3{}^4: 8ABC \cdot C_3 8AD, & y_6 &= C_3{}^5: 8ABC. \end{aligned}$$

These are the parametric equations of a rational F_8 in S_5. The images of points D and E are points on the line $y_1 = y_2 = y_3 = y_4 = 0$.

From the line $y_1 = y_2 = y_3 = y_4 = 0$ as a center project F_8 into the S_3, $y_5 = y_6 = 0$, giving a surface whose parametric equations are

$$(20)\quad \begin{aligned} y_1 &= C_{15}: 8A^5B^4CDE, & y_2 &= C_{12}: 8A^4B^3DE \cdot C_3: 8ABC,\\ y_3 &= C_9: 8A^3B^2DE \cdot C_3{}^2: 8ABC, & y_4 &= C_6: 8A^2BDE \cdot C_3{}^3: 8ABC. \end{aligned}$$

The surface is an F_6. The points D and E go into lines on this surface. The image of C is the line $y_3 = y_4 = 0$. The image of $C_3: 8ABC$ is the point $(1, 0, 0, 0)$.

Since the genus of a member of the system is 5, each plane section of F_6 is a C_6 of genus 5. Hence there is a double C_5 on the surface.

By (15) we see that the section of F_6 by the first polar of a point not on F_6 goes into a composite $C_{75}: 8A^{25}B^{20}C^5D^5E^5$ which consists of the Jacobian $C_{42}: 8A^{14}B^{11}C^2D^2E^2$; $C_3: 8ABC$, the image of the singular point $(0, 0, 0, 1)$; and the image of the double C_5 which is a $C_{30}: 8A^{10}B^8C^2D^3E^3$. A general $C_{30}: 8A^{10}B^8C^2D^2E^2$ goes into a general quadric section of F_6 but the $C_{30}: 8A^{10}B^8C^2D^3E^3$ goes into the quadric section containing the double C_5 with the image lines of points D and E as the residual intersection of the quadric with F_5.

If two surfaces of order n_1 and n_2 respectively contain a C_m of order m and rank r, genus p to multiplicity i_1 and i_2 respectively, the residual C_v meets C_m in t points and has genus π where *

$$(21)\quad \begin{aligned} t &= m(i_2n_1 + i_1n_2 - 2i_1i_2) - i_1i_2r,\\ \pi &= [v(n_1 + n_2 - 4) - (i_1 + i_2 - 1)t]/2 + 1,\\ r &= 2m + 2p - 2. \end{aligned}$$

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* Noether, *Annali di Mathematica*, Ser. 2, Vol. 5 (1871), pp. 163-177.

For the case in question

$$n_1 = 2 \qquad i_1 = 1 \qquad m = 5$$
$$n_2 = 6 \qquad i_2 = 2$$

and, since C_v is a composite conic,

$$v = 2 \text{ and } \pi = -1.$$

Substituting the above values in equations (21) we find that $p = 2$; that is, the double C_5 on F_6 is a curve of genus 2.

A C_5 genus 2 is the partial intersection of a quadric and a cubic surface, the residual being a ruling on the quadric.

Let F_3 be a cubic surface containing the C_5 and one line of the degenerate C_2. Then the number of intersections of the line with C_5 is, by (21), $t = 3$. Hence the line images of points D and E are trisecants of the quintic C_5.

6(a). If B is chosen on a certain locus there is a $C_{12} : 8A^4B^4$ other than $C_3^4 : 8AB$, and we can take as parametric equations of the surface

$$(22) \quad \begin{aligned} y_1 &= C_{12} : 8A^4B^4DE \cdot C_3 : 8AC, \qquad y_2 = C_{12} : 8A^4B^4DE \cdot C_3 : 8ABC, \\ y_3 &= C_9 : 8A^3B^2DE \cdot C_3^2 : 8ABC, \quad y_4 = C_6 : 8A^2BDE \cdot C_3^3 : 8ABC. \end{aligned}$$

The surface is again an F_6 with plane sections of genus 5 and a double curve C_5 of order 5.

The image of C_{12} is the line $y_1 = y_2 = 0$. Since $C_{12} : 8A^4B^4DE$ meets a general $C_{15} : 8A^5B^4CDE$ in 2 residual points, $y_1 = y_2 = 0$ is a double line; hence the double C_5 is composite with $y_1 = y_2 = 0$ as a component.

For a point near B in (x), y_1, y_3 and y_4 are infinitesimals of order four while y_2 is an infinitesimal of order five, hence the image of B is a rational quartic in the plane $y_2 = 0$. The images of D and E are lines meeting the line $y_1 = y_2 = 0$, since D and E are on C_{12} in plane (x).

6(b). Again if B is chosen on a certain locus so that there is a $C_9 : 8A^3B^3$ other than $C_3^3 : 8AB$ we can take the following as parametric equations of a surface

$$(23)^* \quad \begin{aligned} y_1 &= C_9 : 8A^3B^3DEF \cdot C_6 : 8A^2BCDEF, \\ y_2 &= C_9 : 8A^3B^3DEF \cdot C_3 : 8ABC \cdot C_3 : 8A, \\ y_3 &= C_9 : 8A^3B^3DEF \cdot C_3^2 : 8ABC, \\ y_4 &= C_6 : 8A^2BDEF \cdot C_3^3 : 8ABC. \end{aligned}$$

The surface is an F_5 genus 5. Since each plane section is a quintic curve of genus 5 there is a double line on the surface.

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* D. Montesano, *Rendiconti della Reale Accademia di Napoli*, Ser. 3, Vol. 13 (1907), pp. 66-68.

The image of C_9 is the point $(0, 0, 0, 1)$. The image of $C_3: 8AB$ is the point $(1, 0, 0, 0)$. The image of C is the line $y_3 = y_4 = 0$. The images of points D, E and F which are on C_9 are three lines in the plane $y_1 = 0$, passing thru $(0, 0, 0, 1)$, the image of C_9.

The image of C_6 is the line $y_1 = y_4 = 0$. Since $C_6: 8A^2BCDEF$ has, with a general $C_{15}: 8A^5B^4CDEF$, two residual intersections, every plane section of F_5 has a double point on $y_1 = y_4 = 0$, which is, then, the double line on the surface.

A plane section thru the point $(0, 0, 0, 1)$ goes into a composite curve in plane (x) of the form

$$C_9: 8A^3B^3DEF \cdot C_6: 8A^2BC.$$

Two curves of this type meet in two residual points not on C_9, hence the line of intersection of two planes through $(0, 0, 0, 1)$ meets F_5 in two points other than $(0, 0, 0, 1)$, hence $(0, 0, 0, 1)$ is a triple point on F_5.

By the method of (15) we can show that the images of the points D, E and F in (x) form the residual intersection of the plane through the double line and the point $(0, 0, 0, 1)$.

7. *The system* $C_{18}: 8A^6B^5$. This system is of dimension 6 and by a process of reasoning similar to that in the previous cases we obtain the surface in S_3 given parametrically by

$$(24)\quad \begin{aligned} y_1 &= C_{18}: 8A^6B^5CD^2, & y_2 &= C_{15}: 8A^5B^4D^2 \cdot C_3: 8ABC, \\ y_3 &= C_{12}: 8A^4B^3D^2 \cdot C_3{}^2: 8ABC, & y_4 &= C_9: 8A^3B^2D^2 \cdot C_3{}^3: 8ABC, \end{aligned}$$

The surface is an F_6 and each plane section is a C_6 genus 5; hence the surface contains a double C_5 of order 5. The image of the curve C_3 is the point $(1, 0, 0, 0)$. The image of the point C is the line $y_3 = y_4 = 0$. The image of the point D is a conic on F_6.

Through the double C_5 on F_6 passes one quadric surface whose residual intersection with F_6 is a conic C_2.

Again referring to (15) we can show that the double curve C_5 goes into a $C_{36}: 8A^{12}B^{10}C^2D^5$. Since a general $C_{36}: 8A^{12}B^{10}C^2D^4$ goes into a general quadric section of F_6, the $C_{36}: 8A^{12}B^{10}C^2D^5$ goes into the section made by the quadric on the double C_5, which, therefore, has the image conic of point D as the residual intersection with F_6.

7(a). If B is chosen so that there is a $C_9: 8A^3B^3$ other than $C_3{}^3: 8AB$ we obtain a surface whose parametric equations are

$$(25)^* \qquad \begin{aligned} y_1 &= C_9 : 8A^3B^3DE \cdot C_9 : 8A^3B^2CD, \\ y_2 &= C_9 : 8A^3B^3DE \cdot C_6 : 8A^2BD \cdot C_3 : 8ABC, \\ y_3 &= C_9 : 8A^3B^3DE \cdot C_8 : 8AD \cdot C_3{}^2 : 8ABC, \\ y_4 &= C_{12} : 8A^4B^3D^2E \cdot C_3{}^2 : 8ABC. \end{aligned}$$

The surface is an F_5 with plane sections of genus 5; hence there is a double line on the surface. The image of $C_9 : 8A^3B^3$ is the point $(0, 0, 0, 1)$. The image of $C_3 : 8ABC$ is the point $(1, 0, 0, 0)$. The images of D and E are a conic and a line, both of which must pass through the point $(0, 0, 0, 1)$, since D and E are on $C_9 : 8A^3B^3$. The image of C is the line $y_3 = y_4 = 0$.

The plane determined by the double line and the point $(0, 0, 0, 1)$ has a C_3 as residual intersection with F_5.

From (15) we find that the double line goes into a $C_{18} : 8A^6B^5CD^3E^2$. Since a general C_{18} goes into a general plane section of F_5, the $C_{18} : 8A^6B^5CD^3E^2$ goes into the section made by the plane through the double line and the point $(0, 0, 0, 1)$ and contains the images of points D and E as the residual intersection with F_5. The residual C_3 in which the plane determined by the double line and the point $(0, 0, 0, 1)$ meets the F_5 is therefore composite, and consists of a conic and a line, images of D and E.

A section of F_5 by a plane $a_1y_1 + a_2y_2 + a_3y_3 = 0$ goes into a composite curve of the form

$$\begin{aligned} C_9 : 8A^3B^3DE(a_1C_9 : 8A^3B^2CD + a_2C_6 : 8A^2BD \cdot C_3 : 8ABC \\ + a_3C_3 : 8AD \cdot C_3 : 8ABC) = 0; \end{aligned}$$

that is

$$C_9 : 8A^3B^3DE \cdot C_9 : 8A^3B^2CD.$$

Two of these $C_9 : 8A^3B^2CD$ meet in three residual points, hence $(0, 0, 0, 1)$ is a double point on F_5.

7(b). If the point B is so chosen that there is a $C_{15} : 8A^5B^5$ other than $C_3{}^5 : 8AB$, the parametric equations of the surface may be taken as

$$(26) \qquad \begin{aligned} y_1 &= C_{15} : 8A^5B^5DEF \cdot C_3 : 8AC, \qquad y_2 = C_{15} : 8A^5B^5DEF \cdot C_3 : 8ABC, \\ y_3 &= C_{12} : 8A^4B^3DEF \cdot C_3{}^2 : 8ABC, \quad y_4 = C_9 : 8A^3B^2DEF \cdot C_3{}^3 : 8ABC. \end{aligned}$$

This surface differs from (24) only in that the line $y_1 = y_2 = 0$ is a double line on the surface and a component of the double C_9.

7(c). If the point B is so chosen that there is a $C_{12} : 8A^4B^4$ other than $C_3{}^4 : 8AB$ then we obtain a surface whose parametric equations are

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* D. Montesano, *Rendiconti della Reale Accademia di Napoli*, Ser. 3, Vol. 13 (1907), pp. 66-68.

$$(27)\qquad \begin{aligned} y_1 &= C_{12}: 8A^4B^4DEFG \cdot C_6 8A^2BC, \\ y_2 &= C_{12}: 8A^4B^4DEFG \cdot C_3: 8ABC \cdot C_3: 8A, \\ y_3 &= C_{12}: 8A^4B^4DEFG \cdot C_3{}^2: 8ABC, \\ y_4 &= C_9: 8A^3B^2DEFG \cdot C_3{}^3: 8ABC. \end{aligned}$$

The surface is an F_6 with plane sections of genus 6 and contains a double C_4. The image of C_{12} is the point $(0, 0, 0, 1)$. A section made by a plane through this point goes into a curve in plane (x) of the form

$$C_{12}: 8A^4B^4DEFG \cdot C_6: 8A^2BC.$$

Two of these $C_6: 8A^2BC$ meet in two residual points, hence the point $(0, 0, 0, 1)$ is 4-fold on the surface.

The images of D, E, F, and G are lines in the plane $y_1 = 0$ passing through $(0, 0, 0, 1)$. The image of C_6 is the line $y_1 = y_4 = 0$. In plane (x), $C_6: 8A^2BCDEFG$ meets a general $C_{18}: 8A^6B^5CDEFG$ in two residual points, hence $y_1 = y_4 = 0$ is a double line and a component of the double C_4.

From (15) we find that the double C_4 goes into a $C_{36}: 8A^{12}B^{10}C^2D^3E^3F^3G^3$. This C_{36} goes into the quadric section on the double C_4 and the point $(0, 0, 0, 1)$ which has as residual intersection with F_6 the composite C_4 consisting of the four lines, images of the points D, E, F and G.

8. *The system* $C_{21}: 8A^7B^6$. This system is of dimension 7 and by the methods previously employed we obtain the surface whose parametric equations are

$$(28)\qquad \begin{aligned} y_1 &= C_{21}: 8A^7B^6CD^2E, & y_2 &= C_{18}: 8A^6B^5D^2E \cdot C_3: 8ABC, \\ y_3 &= C_{15}: 8A^4B^3D^2E \cdot C_3{}^2: 8ABC, & y_4 &= C_{12}: 8A^3B^2D^2E \cdot C_3{}^3: 8ABC. \end{aligned}$$

The surface is an F_7 with plane sections of genus 6, hence there is a double curve C_9 of order 9 on the surface.

The image of C is the line $y_3 = y_4 = 0$, the image of D a conic and the image of E a line. By (15) we find that the double C_9 goes into a curve in the plane (x) of the form $C_{63}: 8A^{21}B^{18}C^3D^7E^3$. This C_{63} goes into the section of F_7 made by the cubic surface containing the double C_9 and has as residual intersection a composite cubic curve consisting of the conic image of D and the line image of E.

8(a). If B is so chosen that there is a $C_{15}: 8A^5B^5$ other than $C_3{}^5: 8AB$ a surface is determined whose parametric equations are

$$(29)\qquad \begin{aligned} y_1 &= C_{15}: 8A^5B^55D \cdot C_6: 8A^2BC, & y_2 &= C_{15}: 8A^5B^55D \cdot C_3: 8ABC \cdot C_3 8A, \\ y_3 &= C_{15}: 8A^5B^55D \cdot C_3{}^2: 8ABC, & y_4 &= C_{12}: 8A^4B^35D \cdot C_3{}^3: 8ABC. \end{aligned}$$

The surface is an F_7 with plane sections of genus 7 hence there is a double curve C_8 of order 8 on the surface. The image of C is the line $y_3 = y_4 = 0$. The image of $C_3 : 8ABC$ is the point $(0, 0, 0, 1)$. Since a line through $(0, 0, 0, 1)$ meets the surface in only two residual points this point is of order 5 on the surface. The image of B is a rational sextic in the plane $y_3 = 0$ with a point of order 5 at $(0, 0, 0, 1)$.

The images of the D_i $(i = 1, 2, \cdots, 5)$ are five lines on the surface passing through the point $(0, 0, 0, 1)$ and form the residual intersection with F_7 of the cubic surface through the double C_8.

8(b). If the point B is so chosen that there is a $C_{12} 8A^4B^4$ other than $C_3^4 : 8AB$ a surface is determined whose parametric equations are

$$
\begin{aligned}
y_1 &= C_{12} : 8A^4B^4DEF \cdot C_9 : 8A^3B^2CD^2EF, \\
y_2 &= C_{12} : 8A^4B^4DEF \cdot C_6 : 8A^2BD \cdot C_3 : 8ABC, \\
y_3 &= C_{12} : 8A^4B^4DEF \cdot C_3^2 : 8ABC \cdot C_3 : 8AD, \\
y_4 &= C_9 : 8A^3B^2CD^2EF \cdot C_3^4 : 8ABC.
\end{aligned} \tag{30}
$$

The surface is an F_6 with plane sections of genus 6, hence there is a double C_4 on the surface.

The image of C_{12} is the point $(0, 0, 0, 1)$, the image of $C_3 : 8ABC$ the point $(1, 0, 0, 0)$, and the image of C_9 is the line $y_1 = y_4 = 0$. The image of D is a conic in the plane $y_1 = 0$ passing through the point $(0, 0, 0, 1)$. The images of E and F are lines in the plane $y_1 = 0$. The line $y_1 = y_4 = 0$ is a double line. The point $(0, 0, 0, 1)$ is a triple point on the surface.

By the methods employed in the previous cases we find that the residual intersection with F_6 of the quadric through the double C_4 is a composite C_4 consisting of the conic image of the point D and the line images of the points E and F.

9. *Conclusion.* It is now clear that the processes developed in this paper can be carried on indefinitely for any linear system of the type $C_{3n} : 8A^nB^{n-1}$ containing a pencil $C_{3i} : 8A^iB^i$.